TWMS J. Pure Appl. Math. V.12, N.2, 2021, pp.181-192

SOME NEW CLASSES OF STRONGLY GENERALIZED PREINVEX FUNCTIONS

M.A. NOOR[1], K.I. NOOR[1]

Abstract. In this paper, we define and introduce some new concepts of the relative strongly preinvex functions and relative strongly monotone operators with respect to the auxiliary non-negative function and bifunction. We establish some new relationships among various concepts of relative strongly preinvex functions. As special cases, one can obtain various new and known consequences of our results. Results obtained in this paper can be viewed as refinement and improvement of previously known results.

Keywords: preinvex functions, monotone operators, invex functions, strongly preinvex functions.

AMS Subject Classification: 49J40, 90C33.

1. Introduction

In recent years, several extensions and generalizations have been considered for classical convexity. Strongly convex functions were introduced and studied by Polyak [34], which play an important part in the optimization theory and related areas, see, for example, [2, 3, 4, 5, 8, 9, 10, 16, 17, 18, 19, 20] and the references therein. Adamek [1] introduced another class of convex functions with respect to an arbitrary non-negative function, called relative strongly convex functions. With an appropriate choice of non-negative function, one can obtain various known classes of convex functions. For the properties of the relative strongly convex functions, see Adamek [1], Nikodem et al. [2, 5, 7, 8, 9, 10] and Noor [15]. Hanson [7] introduced the concept of invex function for the differentiable functions, which played significant part in the mathematical programming. Ben-Israel and Mond [6] introduced the concept of invex set and preinvex functions. It is known that the differentiable preinvex function are invex functions. The converse also holds under certain conditions, see [20]. Noor [16] proved that the minimum of the differentiable preinvex functions on the invex set can be characterized by a class of variational inequalities, which is known as the variational-like inequality. For the recent developments in variational-like inequalities and invex equilibrium problems, see [16, 17, 19, 29, 31, 32, 33, 35, 36, 37, 38] and the references therein. Noor [20, 21, 22] proved that a function f is preinvex function, if and only if, it satisfies the Hermite-Hadamard type integral inequality. This result has inspired a great deal of subsequent work which has expanded the role and applications of the invexity in nonlinear optimization and engineering sciences. Noor at el. [23, 24, 25, 28, 30] investigated the properties of the strongly preinvex functions and their variant forms.

Inspired by the work of Adamek [1] and Nikodem et al. [2, 5, 8, 9, 10, 14], we introduce and consider another class of nonconvex functions with respect to an arbitrary non-negative function. This class of nonconvex functions is called the relative strongly preinvex functions. Serval new concepts of monotonicity are introduced. We establish the relationship between these classes and derive some new results under some mild conditions. As special cases, on can obtain various

[1]Department of Mathematics, COMSATS University Islamabad, Islamabad, Pakistan
e-mail: noormaslam@gmail.com, khalidan@gmail.com

Manuscript received January 2017.

new and refined versions of known results. It is expected that the ideas and techniques of this paper may stimulate further research in this field.

2. Preliminary results

Let K_η be a nonempty closed set in a real Hilbert space H. We denote by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ be the inner product and norm, respectively. Let $F : K_\eta \rightarrow R$ be a continuous function and let $h : [0,\infty) \rightarrow R$ be a non-negative function.

Definition 2.1. [6]. *The set K_η in H is said to be invex set with respect to an arbitrary continuous bifunction $\eta(\cdot,\cdot) :: K_\eta \times K_\eta \rightarrow R$, if*

$$u + t\eta(v,u) \in K_\eta, \qquad \forall u,v \in K_\eta, t \in [0,1].$$

The invex set K_η is also called η-connected set. Note that the invex set with $\eta(v,u) = v - u$ is a convex set, but the converse is not true. For example, the set $K_\eta = R - (-\frac{1}{2},\frac{1}{2})$ is an invex set with respect to η, where

$$\eta(v,u) = \begin{cases} v-u, & \text{for} \quad v>0, u>0 \quad \text{or} \quad v<0, u<0 \\ u-v, & \text{for} \quad v<0, u>0 \quad \text{or} \quad v<0, u<0. \end{cases}$$

It is clear that K_η is not a convex set.

Remark 2.1. *We would like to emphasize that, if $u + \eta(v,u) = v, \quad \forall u,v \in K_\eta$, then $\eta(v,u) = v - u$. Consequently, the η-invex set reduces to the convex set K. Thus, $K_\eta \subset K$. This implies that every convex set is an invex set, but the converse is not true.*

From now onward K_η is a nonempty closed invex set in H with respect to the bifunction $\eta(\cdot,\cdot)$, unless otherwise specified.

Definition 2.2. *The function F on the invex set K_η is said to be relative strongly preinvex with respect to the bifunction $\eta(\cdot,\cdot)$ and a non-negative function h, if there exists a constant $\mu > 0$, such that*

$$F(u + t\eta(v,u)) \leq (1-t)F(u) + tF(v) - \mu t(1-t)h(\eta(v,u)), \quad \forall u,v \in K_\eta, t \in [0,1].$$

The function F is said to be relative strongly preincave if and only if, $-F$ is relative strongly preinvex. Note that every relative strongly convex function is a relative strongly preinvex, but the converse is not true.
Consequently, we have a new concept of affine preinvex functions.

Definition 2.3. *The function F on the invex set K_η is said to be relative strongly affine preinvex with respect to the bifunction $\eta(\cdot,\cdot)$ and a non-negative function h, if there exists a constant $\mu > 0$, such that*

$$F(u + t\eta(v,u)) = (1-t)F(u) + tF(v) - \mu t(1-t)h(\eta(v,u)), \quad \forall u,v \in K_\eta, t \in [0,1].$$

We now discuss some special cases of the relative strongly preinvex functions.

(I). If $h(\eta(v,u)) = \|\eta(v,u)\|^2$, then the relative strongly preinvex function becomes strongly preinvex functions, that is,

$$F(u + t\eta(v,u)) \leq (1-t)F(u) + tF(v) - \mu t(1-t)\|\eta(v,u)\|^2, \quad \forall u,v \in K_\eta, t \in [0,1].$$

For the properties of the strongly preinvex functions in variational inequalities and equilibrium problems, see Noor [16, 17, 19, 29, 31, 32].

(II). If $\eta(v,u) = v - u$, then the invex set becomes a convex set and preinvex function reduces to the convex function. In this case, Definition 2.2 becomes:

Definition 2.4. [1]. *The function F on the invex set K_η is said to be relative strongly convex with respect to a non-negative function h, if there exists a constant $\mu > 0$, such that*

$$F((1-t)u+tv) \leq (1-t)F(u) + tF(v) - \mu t(1-t)h(v-u), \quad \forall u,v \in K_\eta, t \in [0,1].$$

For the properties and other aspects of the relative strongly functions, see Adamek [1] and Noor [15].

(III). If $\mu = 0$. then Definition 2.2 reduces to:

Definition 2.5. *The function F on the invex set K_η is said to be preinvex, if*

$$F((u+t\eta(v,u)) \leq (1-t)F(u) + tF(v), \quad \forall u,v \in K_\eta, t \in [0,1],$$

which is mainly due to Ben-Isreal and Mond [9].

Definition 2.6. *The function F on the invex set K_η is said to be relative strongly quasi preinvex with respect to a non-negative function h and the bifunction η, if there exists a constant $\mu > 0$ such that*

$$F(u+t\eta(v,u)) \leq \max\{F(u), F(v)\} - \mu t(1-t)h(\eta(v,u)), \quad \forall u,v \in K_\eta, t \in [0,1].$$

Definition 2.7. *The function F on the invex set K_η is said to be relative strongly* log*-preinvex with respect to h and $\eta(\cdot,\cdot)$ in the second sense, if there exists a constant $\mu > 0$ such that*

$$\begin{aligned} \log F(u+t\eta(v,u)) \quad &\leq \quad (1-t)\log F(u)) + t\log F(v) \\ &\quad -\mu t(1-t)h(\eta(v,u)), \quad \forall u,v \in K_\eta, t \in [0,1], \end{aligned}$$

where $F(\cdot) > 0$.

Definition 2.8. *The function F on the invex set K_η is said to be relative strongly* log*-preinvex with respect to h and $\eta(\cdot,\cdot)$, if there exists a constant $\mu > 0$ such that*

$$F(u+t\eta(v,u)) \leq (F(u))^{1-t}(F(v))^t - \mu t(1-t)h(\eta(v,u)), \quad \forall u,v \in K_\eta, t \in [0,1],$$

where $F(\cdot) > 0$.

From the above definition, we have

$$\begin{aligned} F(u+t\eta(v,u)) \quad &\leq \quad (F(u))^{1-t}(F(v))^t - \mu t(1-t)h(\eta(v,u)) \\ &\leq \quad (1-t)F(u) + tF(v) - \mu t(1-t)h(\eta(v,u)) \\ &\leq \quad \max\{F(u), F(v)\} - \mu t(1-t)h(\eta(v,u)), \quad \forall u,v \in K_\eta, t \in [0,1]. \end{aligned}$$

This shows that every relative strongly log-preinvex function is relative strongly preinvex function and every relative strongly preinvev function is a relative quasi-preinvex function. However, the converse is not true.

For $t = 1$, Definition 2.2 and 2.8 reduce to the following condition, which is mainly due to Noor and Noor [8].

Condition A.

$$F(u+t\eta(v,u)) \quad \leq \quad F(v) \qquad \forall v \in K_\eta.$$

For the applications of Condition A, see [2, 7, 10, 11].

Definition 2.9. *The differentiable function F on the invex set K_η is said to be relative strongly invex function with respect to an arbitrary non-negative function h and the bifunction $\eta(\cdot,\cdot)$, if there exists a constant $\mu > 0$ such that*

$$F(v) - F(u) \geq \langle F'(u), \eta(v,u)\rangle + \mu h(\eta(v,u)), \qquad \forall u, v \in K_\eta,$$

where $F'(u))$, is the differential of F at u.

It is noted that, if $\mu = 0$, then the Definition 2.9 reduces to the definition of the invex functions as introduced by Hanson [7]. It is well known that the concepts of preinvex and invex functions play a significant role in the mathematical programming and optimization theory, see [1, 2, 5, 7, 8, 9, 10, 11, 12] and the references therein.

Definition 2.10. *The differentiable function F on the invex set K_η is said to be relative strongly super-quadratic invex function with respect to an arbitrary non-negative function h and the bifunction $\eta(\cdot,\cdot)$, if there exist constants $\mu > 0$ and $\xi \in F'(u)$ such that*

$$F(v) - F(u) \geq \langle \xi, \eta(v,u)\rangle + \mu h(\eta(v,u)), \qquad \forall u, v \in K_\eta.$$

Remark 2.2. *Note that, if $\mu = 0$, then the Definition 2.2-2.8 reduces to the ones in* [6, 8].

Definition 2.11. *An operator $T : K_\eta \to H$ is said to be:*

(1) *relative strongly η-monotone, if and if, there exists a constant $\alpha > 0$ such that*

$$\langle Tu, \eta(v,u)\rangle + \langle Tv, \eta(u,v)\rangle \leq -\alpha\{h(\eta(v,u)) + h(\eta(u,v))\}, \quad u, v \in K_\eta,$$

(2) *η-monotone, if and if,*

$$\langle Tu, \eta(v,u)\rangle + \langle Tv, \eta(u,v)\rangle \leq 0, \quad u, v \in K_\eta,$$

(3) *relative strongly η-pseudomonotone, if and if, there exists a constant $\nu > 0$ such that*

$$\langle Tu, \eta(v,u)\rangle + \nu h(\eta(v,u)) \geq 0 \Rightarrow -\langle Tv, \eta(u,v)\rangle \geq 0, \quad u, v \in K_\eta,$$

(4) *relative strongly relaxed η-pseudomonotone, if and if, there exists a constant $\mu > 0$ such that*

$$\langle Tu, \eta(v,u)\rangle \geq 0 \Rightarrow -\langle Tv, \eta(u,v)\rangle + \mu h(\eta(u,v)) \geq 0, \quad u, v \in K_\eta,$$

(5) *strictly η-monotone, if and if,*

$$\langle Tu, \eta(v,u)\rangle + \langle Tv, \eta(u,v)\rangle < 0, \quad u, v \in K_\eta,$$

(6) *η-pseudomonotone, if and if,*

$$\langle Tu, \eta(v,u)\rangle \geq 0 \Rightarrow \langle Tv, \eta(u,v)\rangle \leq 0, \quad u, v \in K_\eta,$$

(7) *quasi η-monotone, if and if,*

$$\langle Tu, \eta(v,u)\rangle > 0 \Rightarrow \langle Tv, \eta(u,v)\rangle \leq 0, \quad u, v \in K_\eta,$$

(8) *strictly η-pseudomonotone, if and if,*

$$\langle Tu, \eta(v,u)\rangle \geq 0 \Rightarrow \langle Tv, \eta(u,v)\rangle < 0, \quad u, v \in K_\eta.$$

Note that, if $\eta)v,u) = v - u$, then the invex set K_η is a convex set K. This clearly shows that Definition 2.11 is more general than and includes the ones in [7, 8, 9, 10, 11] as special cases.

Definition 2.12. *A differentiable function F on the invex set K_η is said to be relative strongly pseudo η-invex function, if and if, if there exists a constant $\mu > 0$ such that*

$$\langle F'(u), \eta(v,u)\rangle + \mu h(\eta(u,v)) \geq 0 \Rightarrow F(v) - F(u) \geq 0, \quad \forall u, v \in K_\eta.$$

Definition 2.13. *A differentiable function F on K_η is said to be relative strongly quasi-invex function, if and if, if there exists a constant $\mu > 0$ such that*

$$F(v) \leq F(u) \Rightarrow \langle F'(u), \eta(v,u)\rangle + \mu h(\eta(u,v)) \leq 0, \qquad \forall u, v \in K_\eta.$$

Definition 2.14. *The function F on the set K_η is said to be pseudo-invex, if*

$$\langle F'(u), \eta(v,u)\rangle \geq 0 \Rightarrow F(v) \geq F(u), \qquad \forall u, v \in K_\eta.$$

Definition 2.15. *The differentiable function F on the K_η is said to be quasi-invex function, if*

$$F(v) \leq F(u) \Rightarrow \langle F'(u), \eta(v,u)\rangle \leq 0, \qquad \forall u, v \in K_\eta.$$

If $\eta(v,u) = -\eta(v,u), \forall u, v \in K_\eta$, that is, the function $\eta(\cdot,\cdot)$ is skew-symmetric, then Definition 2.11-2.15 reduces to the ones in [9, 10, 11]. This shows that the concepts introduced in this paper represent an improvement of the previously known ones. All these new concepts may play important and fundamental part in the mathematical programming and optimization.

We also need the following assumption regarding the bifunction $\eta(\cdot,\cdot)$, which is due to Mohan and Neogy [11].

Condition C. Let $\eta(\cdot,\cdot) : K_\eta \times K_\eta \to H$ satisfy assumptions

$$\begin{aligned} &\eta(u, u + t\eta(v,u)) = -t\eta(v,u), \\ &\eta(v, u + t\eta(v,u)) = (1-t)\eta(v,u), \quad \forall u, v \in K_\eta, t \in [0,1]. \end{aligned}$$

Clearly for $t = 0$, we have $\eta(u,v) = 0$, if and only if $u = v, \forall u, v \in K_\eta$. One can easily show [12, 13] that $\eta(u + t\eta(v,u), u) = t\eta(v,u), \forall u, v \in K\eta$.

3. Main results

In this section, we consider some basic properties of relative strongly preinvex functions on the invex set K_η. Through out this section we assume that the non-negative function h is even and homogeneous of degree two, that is,
$h(-u) = h(u), \quad h(\gamma u) = \gamma^2 h(u), \quad \forall u \in H, \gamma \in R$, unless otherwise specified.

Theorem 3.1. *Let F be a differentiable function on the invex set K_η in H and let Condition C hold. Then the function F is relative strongly preinvex function, if and only if, F is a relative strongly invex function.*

Proof. Let F be a relative strongly preinvex function on the invex set K_η. Then

$$F(u + t\eta(v,u)) \leq (1-t)F(u) + tF(v) - t(1-t)\mu h(\eta(v,u)), \quad \forall u, v \in K_\eta,$$

which can be written as

$$F(v)-F(u)\geq\{\frac{F(u+t\eta(v,u))-F(u)}{t}\}+(1-t)\mu h(\eta(v,u)).$$

Taking the limit in the above inequality as $t\to 0$, we have

$$F(v)-F(u)\geq\langle F'(u),\eta(v,u))\rangle+\mu h(\eta(v,u)).$$

This shows that F is a relative strongly invex function.

Conversely, let F be a relative strongly invex function on the invex set K. Then $\forall u,v\in K_\eta, t\in[0,1]$, $v_t=u+t\eta(v,u)\in K_\eta$ and using Condition C, we have

$$\begin{aligned} & F(v)-F(u+t\eta(v,u)) \\ \geq\ & \langle F'(u+t\eta(v,u)),\eta(v,u+t\eta(v,u))\rangle+\mu h(\eta(v,u+t\eta(v,u))) \\ =\ & (1-t)\langle F'(u+t\eta(v,u)),\eta(v,u)\rangle+\mu(1-t)^2h(\eta(v,u)). \end{aligned} \tag{1}$$

In a similar way, we have

$$\begin{aligned} & F(u)-F(u+t\eta(v,u)) \\ \geq\ & \langle F'(u+t\eta(v,u)),\eta(u,u+t\eta(v,u))\rangle+\mu h(\eta(u,u+t\eta(v,u))) \\ =\ & -t\langle F'(u+t\eta(v,u)),\eta(v,u)\rangle+\mu t^2h(\eta(v,u)). \end{aligned} \tag{2}$$

Multiplying (1) by t and (2) by $(1-t)$ and adding the resultant, we have

$$F(u+t\eta(v,u))\leq(1-t)F(u)+tF(v)-t(1-t)\mu h(\eta(v,u)),$$

showing that F is a relative strongly preinvex function. $\square$

Theorem 3.2. *Let F be differentiable on the invex set K_η. Let Condition A and Condition C Hold. The F is a relative strongly invex function, if and only if, $F'(.)$ is relative strongly η-monotone.*

Proof. Let F be a relative strongly invex function on the invex set K_η. Then

$$F(v)-F(u)\geq\langle F'(u),\eta(v,u))\rangle+\mu h(\eta(v,u))\quad\forall u,v\in K_\eta. \tag{3}$$

Changing the role of u and v in (3), we have

$$F(u)-F(v)\geq\langle F'(v),\eta(u,v)\rangle+\mu h(\eta(u,v))\quad\forall u,v\in K_\eta. \tag{4}$$

Adding (3) and (4), we have

$$\langle F'(u),\eta(v,u))\rangle+\langle F'(v),\eta(u,v)\rangle\geq-\mu\{h(\eta(v,u))+h(\eta(u,v))\}, \tag{5}$$

which shows that F' is relative strongly η-monotone.

Conversely, let $F'(.)$ be relative strongly η-monotone. From (5), we have

$$\langle F'(v),\eta(u,v)\rangle\geq\langle F'(u),\eta(v,u))\rangle-\{h(\eta(v,u))+h(\eta(u,v))\}, \tag{6}$$

Since K is an invex set, $\forall u,v\in K_\eta$, $t\in[0,1]$ $v_t=u+t\eta(v,u)\in K_\eta$. Taking $v=v_t$ in (6) and using Condition C, we have

$$\begin{aligned}\langle F'(v_t),\eta(u,u+t\eta(v,u))\rangle &\leq \langle F'(u),\eta(u+t\eta(v,u),u))\rangle-\mu\{h(\eta(u+t\eta(v,u),u)) \\ &+ h(\eta(u,u+t\eta(v,u)))\} \\ &= -t\langle F'(u),\eta(v,u)\rangle-2t^2\mu h(\eta(v,u)),\end{aligned}$$

which implies that

$$\langle F'(v_t), \eta(v,u)\rangle \geq \langle F'(u), \eta(v,u) + 2\mu th(\eta(v,u)). \tag{7}$$

Let $g(t) = F(u + t\eta(v,u))$. Then from (7), we have

$$\begin{aligned} g'(t) &= \langle F'(u + t\eta(v,u)), \eta(v,u)\rangle \\ &\geq \langle F'(u), \eta(v,u) + 2\mu th(\eta(v,u)). \end{aligned} \tag{8}$$

Integrating (8) between 0 and 1, we have

$$g(1) - g(0) \geq \langle F'(u), \eta(v,u) + \mu h(\eta(v,u)).$$

that is,

$$F(u + t\eta(v,u)) - F(u) \geq \langle F'(u), \eta(v,u) + \mu h(\eta(v,u)).$$

By using Condition A, we have

$$F(v) - F(u) \geq \langle F'(u), \eta(v,u) + \mu h(\eta(v,u)).$$

which shows that F is relative strongly invex function on the invex set K_η. □

From Theorem 3.1 and Theorem 3.2, we have

relative strongly preinvex function $F \Rightarrow$ relative strongly invex function $F \Rightarrow$ relative strongly η-monotonicity of the differential F' and conversely, if Conditions A and C hold.

We now give a necessary condition for strongly η-pseudo-invex function.

Theorem 3.3. *Let $F'(.)$ be relative strongly relaxed η- pseudomonotone and Condition A and C hold. Then F is a relative strongly η-pseudo-invex function.*

Proof. Let $F'(.)$ be relative strongly relaxed η-pseudomonotone. Then, $\forall u, v \in K_\eta$,

$$\langle F'(u), \eta(v,u)\rangle \geq 0.$$

implies that

$$-\langle F'(v), \eta(u,v)\rangle \geq \alpha h(\eta(u,v)). \tag{9}$$

Since K is an invex set, $\forall u, v \in K_\eta$, $t \in [0,1]$, $v_t = u + t\eta(v,u) \in K_\eta$. Taking $v = v_t$ in (9) and using condition Condition C, we have

$$-\langle F'(u + t\eta(v,u)), \eta(u,v)\rangle \geq t\alpha h(\eta(v,u)). \tag{10}$$

Let

$$\xi(t) = F(u + t\eta(v,u)), \quad \forall u, v \in K_\eta, t \in [0,1].$$

Then, using (10), we have

$$\xi'(t) = \langle F'(u + t\eta(v,u)), \eta(u,v)\rangle \geq t\alpha h(\eta(v,u)).$$

Integrating the above relation between 0 to 1, we have

$$\xi(1) - \xi(0) \geq \frac{\alpha}{2} h(\eta(v,u)),$$

that is,

$$F(u + t\eta(v,u)) - F(u) \geq \frac{\alpha}{2} h(\eta(v,u)),$$

which implies, using Condition A,

$$F(v) - F(u) \geq \frac{\alpha}{2} h(\eta(v,u)),$$

showing that F is a relative strongly η-pseudo-invex function. $\square$

As special cases of Theorem 3.3, we have the following:

Theorem 3.4. *Let the differentiable $F'(.)$ of a function $F(,)$ on the invex set K be η-pseudomonotone. If Conditions A and C hold, then F is pseudo η-invex function.*

Theorem 3.5. *Let the differential $F'(.)$ of a function $F(.)$ on the invex set K be relative strongly η-pseudomonotone. If Conditions A and C hold, then F is relative strongly pseudo η-invex function.*

Theorem 3.6. *Let the differential $F'(.)$ of a function $F(.)$ on the invex set K_η be relative strongly η-pseudomonotone. If Conditions A and C hold, then F is relative strongly pseudo η-invex function.*

Theorem 3.7. *Let the differential $F'(.)$ of a function $F(.)$ on the invex set K_η be η-pseudomonotone. If Conditions A and C hold, then F is pseudo invex function.*

Theorem 3.8. *Let the differential $F'(.)$ of a differentiable preinvex function $F(.)$ be Lipschitz continuous on the invex set K_η with a constant $\beta > 0$. Then*

$$F(u+\eta(v,u)) - F(u) \leq \langle F'(u), \eta(v,u)\rangle + \frac{\beta}{2}\|\eta(v,u)\|^2, \quad u,v \in K_\eta.$$

Proof. Let K be an invex set. Then $\forall u,v \in K_\eta$, $t \in [0,1]$, $u + t\eta(v,u) \in K_\eta$. Now we consider the function

$$\phi(t) = F(u+t\eta(v,u)) - F(u) - t\langle F'(u), \eta(v,u)\rangle.$$

from which it follows that $\phi(0) = 0$ and

$$\phi'(t) = \langle F'(u+t\eta(v,u)), \eta(v,u)\rangle - \langle F'(u), \eta(v,u)\rangle. \tag{11}$$

Integrating (13) between 0 to 1, we have

$$\begin{aligned}
\phi(1) &= F(u+\eta(v,u)) - F(u) - \langle F'(u), \eta(v,u)\rangle \\
&\leq \int_0^1 \mid \phi'(t) \mid \mathrm{d}t \\
&= \int_0^1 \mid F'(u+t\eta(v,u)), \eta(v,u)\rangle - \langle F'(u), \eta(v,u)\rangle \mid \mathrm{d}t \\
&\leq \beta \int_0^1 t\|\eta(v,u)\|^2 \mathrm{d}t \\
&= \frac{\beta}{2}\|\eta(v,u)\|^2,
\end{aligned}$$

which implies that

$$F(u+\eta(v,u)) - F(u) \leq \langle F'(u), \eta(v,u)\rangle + \frac{\beta}{2}\|\eta(v,u)\|^2. \tag{12}$$

$\square$

Definition 3.1. *The function F is said to be sharply relative strongly pseudo preinvex, if there exists a constant $\mu > 0$ such that*

$$\langle F'(u), \eta(v,u)\rangle \geq 0 \Rightarrow F(v) \geq F(v+t\eta(v,u)) + \mu t(1-t)h(\eta(v,u)) \quad \forall u,v \in K_\eta, t \in [0,1].$$

Theorem 3.9. *Let F be a sharply relative strongly pseudo preinvex function on K_η with a constant $\mu > 0$. Then*

$$-\langle F'(v), \eta(v,u)\rangle \geq \mu h(\eta(v,u)) \quad \forall u, v \in K_\eta.$$

Proof. Let F be a sharply relative strongly pseudo preinvex function on K_η. Then

$$F(v) \geq F(v + t\eta(v,u)) + \mu t(1-t)h(\eta(v,u)), \quad \forall u, v \in K_\eta, t \in [0,1].$$

from which we have

$$\frac{F(v + t\eta(v,u)) - F(v)}{t} + \mu t(1-t)h(\eta(v,u)) \leq 0.$$

Taking limit in the above inequality, as $t \to 0$, we have

$$-\langle F'(v), \eta(v,u)\rangle \geq \mu h(\eta(v,u)),$$

the required result. □

Definition 3.2. *A function F is said to be a pseudo preinvex function, if there exists a strictly positive bifunction $W(.,.)$, such that*

$$\begin{aligned} & F(v) < F(u) \\ & \Rightarrow \\ & F(u + t\eta((v,u)) < F(u) + t(t-1)W(v,u), \forall u, v \in K_\eta, t \in [0,1]. \end{aligned}$$

Theorem 3.10. *If the function F is higher order strongly convex function such that $F(v) < F(u)$, then the function F is higher order strongly pseudo preinvex.*

Proof. Since $F(v) < F(u)$ and F is higher order strongly preinvex function, then $\forall u, v \in K_\eta, \quad t \in [0,1]$, we have

$$\begin{aligned} F(u + \eta(v,u)) &\leq F(u) + t(F(v) - F(u)) - \mu t(1-t)h(\eta(v,u)) \\ &< F(u) + t(a-t)(F(v) - F(u)) - \mu t(1-t)h(\eta(v,u)) \\ &= F(u) + t(t-1)(F(u) - F(v)) - \mu t(1-t)h(\eta(v,u)) \\ &< F(u) + t(t-1)W(u,v) - \mu t(1-t)h(\eta(v,u)), \forall u, v \in K_\eta, \end{aligned}$$

□

where $W(u,v) = F(u) - F(v) > 0$, the required result.

We now discuss the optimality for the differentiable generalized strongly preinvex functions, which is the main motivation of our next result.

Theorem 3.11. *Let F be a differentiable higher order strongly preinvex function with modulus $\mu > 0$. If $u \in K_\eta$ is the minimum of the function F, then*

$$F(v) - F(u) \geq \mu h(\eta(v,u)), \quad \forall u, v \in K_\eta. \tag{13}$$

Proof. Let $u \in K_\eta$ be a minimum of the function F. Then

$$F(u) \leq F(v), \forall v \in K_\eta. \tag{14}$$

Since K is an invex set, so, $\forall u, v \in K_\eta, \quad t \in [0,1]$,

$$v_t = u + t\eta(v,u) \in K_\eta.$$

Taking $v = v_t$ in (14), we have

$$0 \leq \lim_{t\to 0}\{\frac{F(u+t\eta(v,u))-F(u)}{t}\} = \langle F'(u), \eta(v,u)\rangle. \tag{15}$$

Since F is differentiable higher order strongly preinvex function, so

$$F(u+t\eta(v,u)) \leq F(u) + t(F(v)-F(u)) - \mu t(1-t)h(\eta(v,u)), \forall u, v \in K_\eta,$$

from which, using (15), we have

$$\begin{aligned} F(v) - F(u) &\geq \lim_{t\to 0}\{\frac{F(u+t\eta(v,u))-F(u)}{t}\} + \mu t(1-t)h(\eta(v,u)) \\ &= \langle F'(u), \eta(v,u)\rangle + \mu h(\eta(v,u)), \end{aligned}$$

the required result (13). □

Remark 3.1. *We would like to mention that, if*

$$\langle F'(u), \eta(v,u)\rangle + \mu h(\eta(v,u)) \geq 0, \quad \forall u, v \in K_\eta, \tag{16}$$

then $u \in K_\eta$ is the minimum of the function F.

The inequality of the type (16) is called the strongly variational-like inequality. In many applications, the strongly variational-like inequality may not arise as the optimality conditions of the differentiable strongly preinvex functions. These facts motivated to introduce a more general problem of which (16) is a special case. To be more precise, for given nonlinear operator T and an arbitrary bifunction $\eta(.,.)$, consider the problem of finding $u \in K_\eta$ such that

$$\langle Tu, \eta(v,u)\rangle + \mu h(\eta(v,u)) \geq 0, \quad \forall u, v \in K_\eta, \tag{17}$$

which is called the strongly variational-like inequality. Using the auxiliary principle technique as developed by Noor [19, 20], one can suggest and investigate various iterative methods for solving strongly variational-like inequalities.

Conclusion

In this paper, we have introduced and studied a new class of preinvex functions with respect to any arbitrary function and bifunction. It is shown that several new classes of strongly preinvex and convex functions can be obtained as special cases of these relative strongly preinvex functions. We have studied the basic properties of these functions. The optimality conditions of the differentiable higher order preinvex functions are characterized by a class of higher order variational-like inequalities. It is an interesting problem to develop some efficient and implementable numerical techniques for solving the variational-like inequalities. For recent generalizations and applications of the preinvex functions and their variant forms, see Noor and Noor [26, 27, 33] and the references therein It is expected that the ideas and techniques of this paper may motivate further research.

Acknowledgements

We wish to express our sincere gratitude to our teachers, students, colleagues, collaborators and friends, who have direct or indirect contributions in the process of this paper. Authors are grateful to the referees for their valuable suggestions and comments, which influenced the final version of this paper.

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